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STRUCTURES DETERMINED BY PRIME IDEALS OF RINGS OF FUNCTIONS

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Introduction. Let $\mathscr{R}$ and $\mathscr{T}_{3\frac{1}{2}}$ respectively denote the category of commutative rings with unity and the category of completely regular Hausdorff spaces; also, $\mathscr{C}$ denotes the full-subcategory of $\mathscr{T}_{3\frac{1}{2}}$ whose objects are compact, and $\mathscr{C}_{TD}$ the full-subcategory of $\mathscr{C}$ whose objects are totally-disconnected. The collection of prime ideals of $A \in \mathscr{R}$ is the underlying set for an object $K_A \in \mathscr{C}_{TD}$ and K is contravariantly functorial. If C denotes the contravariant functor which assigns to each $X \in \mathscr{T}_{3\frac{1}{2}}$ the ring $C(X)$ of real-valued continuous functions on X, then the resulting functor K_C is the domain of a natural transformation $\tilde{\theta}\colon K_C \to \beta$, where β denotes the Stone-Čech reflection of $\mathscr{T}_{3\frac{1}{2}}$ into $\mathscr{C}$. The prime z-ideals of $C(X)$ also furnish such a space ζX, functor ζ and natural transformation $\theta\colon \zeta \to \beta$. In the appropriate category, ζ fills in a diagram which exhibits β as a push-out.

Topological properties of $K_{C(X)}$ and ζX are studied and ζX is characterized as a certain compactification of X_P, the $\boldsymbol{P}$-topology on X, which helps establish the place of ζX between βX_P and βX.

The above results are applied in an investigation of the continuous and order-preserved image of ζY in both of ζX and βX arising from $f\colon Y \to X$. As one consequence, the prime z-ideal structure and the minimal prime ideal structure associated with X is illuminated by the corresponding structures associated with certain subspaces of X; as another, a convenient simplification and unification is provided for approaching several types of problems found in the literature on prime ideals of $C(X)$.

1. **The functors K_C and ζ.** If $A \in \mathscr{R}$, then K_A denotes the collection of prime ideals of A (which, in our usage, does not include A itself). The following notations are used for a subset $S \subseteq A$ and $a \in A$: $\tilde{h}(S) = \{P \in K_A : S \subseteq P\}$, $\tilde{h}^c(S) = K_A \backslash \tilde{h}(S)$, $\tilde{h}(a) = \tilde{h}(\{a\})$ and $\tilde{h}^c(a) = \tilde{h}^c(\{a\})$. One can view K_A as a subset of the product space 2^A, where $2 = \{0, 1\}$ is the two-point discrete space; hereafter K_A is endowed with its topology relative to 2^A. Observing that $\tilde{h}(a) = \pi_a^{\leftarrow}[1] \cap K_A$ and $\tilde{h}^c(a) = \pi_a^{\leftarrow}[0] \cap K_A$, it follows that the collection $\{\tilde{h}(a) : a \in A\} \cup \{\tilde{h}^c(a) : a \in A\}$ is a

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subbase for K_A; moreover, since $\tilde{h}(a_1) \cap \ldots \cap \tilde{h}(a_n) = \tilde{h}(\{a_1, \ldots, a_n\})$ and $\tilde{h}^c(a_1) \cap \cdots \cap \tilde{h}^c(a_n) = \tilde{h}^c(a_1 \cdot \cdots \cdot a_n)$, we see that the collection $\{\tilde{h}(S) \cap \tilde{h}^c(a) : S$ is a finite subset of A and $a \in A\}$ is a base for K_A.

If S is a subset of A which is not a prime ideal, then S fails to satisfy one of a finite number of properties such as $a \in A$, $b \in S$ implies $ab \in S$; that is, there exist $a, b \in A$ with $S \in U = \pi_b^{\leftarrow}[1] \cap \pi_{ab}^{\leftarrow}[0]$. Whichever property does not hold for S, the corresponding U provides a neighborhood of S in 2^A which misses K_A. Hence, K_A is closed in 2^A and $K_A \in \mathscr{C}_{TD}$.

The preceding argument clearly applies to commonly encountered collections of subsets in many concrete categories which are algebraic in nature; for more detail in this direction see [M_3].

K_A deserves to be compared to Spec A, the well-known [B] topology whose underlying set coincides with that of K_A. Since Spec A has the collection of $\tilde{h}^c(a)$ $(a \in A)$ as a base, it is coarser than K_A; a fact which not only yields the known compactness of Spec A but also shows that $K_A = \text{Spec } A$ if and only if every prime ideal of A is maximal (as this is precisely when Spec A is Hausdorff). Spec is contravariantly functorial; specifically, Spec α: Spec $B \to$ Spec A is defined by $P \mapsto \alpha^{\leftarrow}[P]$ for $\alpha: A \to B$ in $\mathscr{R}$ and $P \in \text{Spec } B$. Since Spec α is continuous by virtue of the equality $(\text{Spec } \alpha)^{\leftarrow}[\tilde{h}(a)] = \tilde{h}(\alpha(a))$, it easily follows that $K_\alpha: K_B \to K_A$ is also continuous, where K_α pointwise agrees with Spec α. Using the functorial properties of Spec, we have the following

1.1 Lemma. *K is a contravariant functor from $\mathscr{R}$ to $\mathscr{C}_{TD}$.* □

Further relationships between K_A and Spec A are found in [M_3]; for present purposes, however, we immediately specialize to the case $A = C(X)$. Henceforth, an unmodified X carries the assumption that $X \in \mathscr{T}_{3\frac{1}{2}}$; also, $f: Y \to X$ indicates that f is a continuous function from $Y \in \mathscr{T}_{3\frac{1}{2}}$ to $X \in \mathscr{T}_{3\frac{1}{2}}$. Notations not otherwise defined are found in [GJ].

Since $C: \mathscr{T}_{3\frac{1}{2}} \to \mathscr{R}$ is contravariantly functorial, we have the functor $K_C: \mathscr{T}_{3\frac{1}{2}} \to \mathscr{C}_{TD}$; specifically, for $f: Y \to X$ and $P \in K_{C(Y)}$,

1.2 $K_{C(f)}(P) = \{g \in C(X): g \circ f \in P\}$.

If $f: Y \to X$ and $g \in C(X)$, the obvious equality

1.3 $f^{\leftarrow}[Z(g)] = Z(g \circ f)$

is repeatedly used; in particular, it justifies (b) and (c) of the following proposition.

1.4 Proposition. *K_C is a functor from $\mathscr{T}_{3\frac{1}{2}}$ to $\mathscr{C}_{TD}$. Moreover, if $f: Y \to X$, then:*

(a) *$K_{C(f)}$ preserves the ordering of set-inclusion;*

(b) [GJ_1, 2.2] *$K_{C(f)}(P)$ is a (prime) z-ideal of $C(X)$ if P is a prime z-ideal of $C(Y)$;*

(c) *$K_{C(f)}(M_y) = M_{f(y)}$ for $y \in Y$.* □

The space of our major interest is ζX, the collection of prime z-ideals of $C(X)$ topologized relative to $K_{C(X)}$. We let $\hat{f}$ denote $K_{C(f)}$ restricted to ζY if $f: Y \to X$;

1.4(b) permits us to treat $\hat{f}$ as $\hat{f}: \zeta Y \to \zeta X$. If $S \subseteq C(X)$ and $g \in C(X)$, then $\hbar(S) = \tilde{\hbar}(S) \cap \zeta X$; $\hbar^c(S)$, $\hbar(g)$ and $\hbar^c(g)$ are similarly defined.

1.5 Proposition. *The mappings $\zeta: X \mapsto \zeta X$ and $\zeta: f \mapsto \hat{f}$ define a functor ζ from $\mathscr{T}_{3\frac{1}{2}}$ to $\mathscr{C}_{TD}$. Moreover, if $f: Y \to X$, then*

(a) *$\hat{f}$ is order-preserving;*

(b) *$\hat{f}(M_y) = M_{f(y)}$ for $y \in Y$.* □

2. **Some topological properties of ζX and $K_{C(X)}$.** It is straightforward to verify that the category of P-spaces is coreflective in $\mathscr{T}_{3\frac{1}{2}}$; specifically, let $X_P = \boldsymbol{P}(X)$ be the set X topologized by taking the collection $Z(X)$ of zero-sets of X as a base, and let $f_P = \boldsymbol{P}(f): Y_P \to X_P$ pointwise agree with $f: Y \to X$. Then X_P is a P-space and the obvious map $p_X: X_P \to X$ is continuous; in fact, $p = \{p_X\}_{X \in \mathscr{T}_{3\frac{1}{2}}}$ is the natural transformation from the coreflection functor $\boldsymbol{P}$ to $I_{3\frac{1}{2}}$, the identity functor on $\mathscr{T}_{3\frac{1}{2}}$. Moreover, f_P is the unique continuous function $Y_P \to X_P$ with the property $p_X \circ f_P = f \circ p_Y$.

Note that since $Z(X)$ is closed under finite (even countable) intersections, the collection $\{H_{g,h}: g, h \in C(X)\}$ is a base for ζX, where $H_{g,h} = \hbar(g) \cap \hbar^c(h)$.

2.1 Theorem. *ζX is a compactification of X_P. Specifically, the map $\nu_X: X_P \to \zeta X$ defined by $x \mapsto M_x$ ($x \in X_P$) is a homeomorphism onto the collection $\mathscr{M}_F(X)$ of fixed maximal ideals which is dense in ζX.*

Proof. Since $Z(X)$ is a base for the space X_P which is coarser than X and $\nu_X^{\leftarrow}[H_{g,h}] = Z(g) \cap \operatorname{coz}(h)$, it follows that the one-to-one correspondence ν_X of X_P onto $\mathscr{M}_F(X)$ is continuous; it is also open as $\nu_X[Z(g)] = \hbar(g) \cap \mathscr{M}_F(X)$. Moreover, if $\varnothing \neq H_{g,h}$, then $Z(g) \nsubseteq Z(h)$ and $M_x \in H_{g,h}$ for each $x \in Z(g) \backslash Z(h)$. □

Let $f: Y \to X$. Since $\nu_Y: Y_P \cong \mathscr{M}_F(Y)$ is dense in ζY, 1.5(b) shows that $\hat{f}$ is the unique extension of f_P to ζY; consequently, we sometimes refer to $\hat{f}$ as *the ζ-extension* of f to ζY. If Y is a P-space, then $\zeta Y = K_{C(Y)} = \beta Y$; also, certain Stone-extensions are ζ-extensions and hence have explicit (1.2) representations. One such extension is singled out in the following proposition. Henceforth the notation $\bar{f}$ will be reserved for the Stone-extension of $f: Y \to X$.

2.2 Proposition. *If $f: Y \to X$, then $\hat{f} \circ \nu_Y = \nu_X \circ f_P$; that is, $\nu = \{\nu_X\}_{X \in \mathscr{T}_{3\frac{1}{2}}}$ is a natural transformation from $\boldsymbol{P}$ to ζ. Moreover, $\bar{\nu}_X = \hat{p}_X$; hence,*

$$\bar{\nu}_X(M) = \{g \in C(X): Z(g) \in Z[M]\}$$

for $M \in \beta X_P$, and $\bar{\nu}_X(M_1) = \bar{\nu}_X(M_2)$ if and only if $Z(X) \cap Z[M_1] = Z(X) \cap Z[M_2]$.

Proof. The first statement follows from 1.5(b); the second holds as $\bar{\nu}_X$ and $\hat{p}_X$ each map $\zeta X_P = \beta X_P$ into ζX and agree on dense X_P. □

Note that $\bar{\nu}_X$ is onto ζX (as X_P is dense in ζX) and so is a quotient map; in fact, 2.2 shows that ζX is the quotient of βX_P under the equivalence relation: $M_1 \sim M_2 \Leftrightarrow Z[M_1] \cap Z(X) = Z[M_2] \cap Z(X)$.

We call a point $x \in X$ a *zero-set point* of X if $\{x\} \in Z(X)$. Note that x is a zero-set point of X if and only if x is an isolated point of X_P.

A subspace of a P-space is itself a P-space [GJ, 4L] and a compact P-space is finite [GJ, 4K]. It follows that a P-space is locally compact if and only if it is discrete; also, it is σ-compact if and only if it is countable.

2.3 COROLLARY.

(a) *X_P is open in ζX if and only if either one of the following two (equivalent) conditions holds:*

(1) *X_P is discrete;*

(2) *every point of X is a zero-set point.*

(b) *X_P is a cozero-set of ζX if and only if either one of the following two (equivalent) conditions holds:*

(1) *X_P is countable and discrete (that is, $X_P \cong N$);*

(2) *X is countable and every point of X is a zero-set point.*

(c) *The isolated points of ζX are precisely those of X_P; that is, P is an isolated point of ζX if and only if $P = M_x \in \mathcal{M}_F(X)$ and x is a zero-set point of X.*

Proof. A space Y is open in any compactification $\bar{Y}$ if and only if it is locally compact [GJ, 3.15]; and is a cozero-set of $\bar{Y}$ if and only if it is locally compact and σ-compact (the latter condition from [GJ, 1.10]). Moreover, the isolated points of Y and $\bar{Y}$ coincide [GJ, 3.15]. □

The isolated points of $K_{C(X)}$ are in one-to-one correspondence with the isolated points of X. Unfortunately, $K_{C(X)}$ has no convenient dense subset (as does ζX) and, before proceeding, we introduce the function $\tilde{\theta}_X\colon K_{C(X)} \to \beta X$ defined by $\tilde{\theta}_X(P) = p$, where p is the unique [GJ, 7.15] point in βX with $P \subseteq M^p$. It is shown in [HJ, 5.3] that $\tilde{\theta}_X$ restricted to the minimal prime ideals topologized relative to Spec $C(X)$ is continuous. The proof used there, however, readily adapts to show that $\tilde{\theta}_X$ is continuous when the domain space is Spec $C(X)$; since $K_{C(X)}$ is finer than Spec $C(X)$, we have the continuity of $\tilde{\theta}_X\colon K_{C(X)} \to \beta X$.

2.4 THEOREM. *A point P of $K_{C(X)}$ is isolated if and only if $P = M_x \in \mathcal{M}_F(X)$ and x is an isolated point of X.*

Proof. Following [GJ, 14.2] we call a point in $K_{C(X)}$ an *upper ideal* if it has an immediate predecessor (in the ordering of set inclusion). Similarly, it is a *lower ideal* if it has an immediate successor.

Now let Q be an isolated point of $K_{C(X)}$ and suppose the following two claims hold:

(a) If Q is not maximal, then it is a lower ideal.

(b) If Q is not minimal, then it is an upper ideal.

Then, if $Q \notin \zeta X$, it is neither maximal nor minimal [GJ, 14.7] and so is both an upper and a lower ideal ((a) and (b)), which is impossible [GJ, 14.18]. Hence, $Q \in \zeta X$ and so is isolated there; that is, $Q = M_x \in \mathcal{M}_F(X)$ and x is a zero-set point of X (2.3(c)). Since no prime z-ideal is an upper ideal [GJ, 14.10], Q is also a minimal prime ideal (b); that is, x is a P-point of X. At the same time, being a zero-set point, x is a G_δ-point of X. The conclusion now follows as every P-point G_δ-point is isolated [GJ, 4L]. It remains to establish the claims.

Since $\{Q\}$ is open in $K_{C(X)}$, there exist $g_1, \ldots, g_n$, $h \in C(X)$ with $\{Q\} = \{P \in K_{C(X)} : g_1, \ldots, g_n \in P$ and $h \notin P\}$. In what follows, the symbol $\subset$ denotes proper inclusion.

(a) If $Q \subset P \in K_{C(X)}$, then $g_1, \ldots, g_n \in P$ and so $h \in P$ (otherwise $P = Q$). Let $\bar{Q} = \bigcap_{Q \subset P} P$, then $\bar{Q} \in K_{C(X)}$ [GJ, 14.2(a) and 14.3(c)] and is clearly the desired immediate successor of Q.

(b) Choose $Q_0 \subset Q$ and let $\underline{Q} = \bigcup_{Q_0 \subseteq P \subset Q} P$; then $\underline{Q} \in K_{C(X)}$ (as with $\bar{Q}$) and we have only to show that $\underline{Q} \neq Q$. Suppose $\underline{Q} = Q$, then for each $k = 1, \ldots, n$ there is $P_k \in K_{C(X)}$ with $g_k \in P_k$ and $Q_0 \subseteq P_k \subset Q$. Since the prime ideals containing Q_0 form a chain [GJ, 14.3(c)], we can choose $k_0 \in \{1, \ldots, n\}$ with $P_k \subseteq P_{k_0}$ for every $k = 1, \ldots, n$; hence, $g_1, \ldots, g_n \in P_{k_0}$. But $h \notin P_{k_0}$ (otherwise $h \in Q$) and so $P_{k_0} = Q$; a contradiction. We have now established the necessity of 2.4.

If x is an isolated point of X, then it is a P-point [GJ, 4L] and so $\{M_x\} = \tilde{\theta}_X^{\leftarrow}[x]$ is open in $K_{C(X)}$. □

It follows from Stone's Representation Theorem for Boolean rings that a space X is in $\mathscr{C}_{TD}$ if and only if $X = K_A$ for some $A \in \mathscr{R}$; that is, the contravariant functor $K: \mathscr{R} \to \mathscr{C}_{TD}$ is onto. The following corollary shows that neither K_C nor ζ is onto $\mathscr{C}_{TD}$ and so raises the (as yet unsolved) problem of characterizing those $X \in \mathscr{C}_{TD}$ which are a $K_{C(Y)}$, or a ζY, for some $Y \in \mathscr{T}_{3\frac{1}{2}}$.

2.5 Corollary. *Let X have at least two points. Then $K_{C(X)}$, or ζX, is the one-point compactification of a discrete space if and only if X is finite.*

Proof. The sufficiency is obvious (X is assumed to have at least two points).

(a) Let $K_{C(X)}$ be the one-point compactification of its discrete subspace Y with $P^* \in K_{C(X)}$ the point-at-infinity. Then each point $P \in Y$ is fixed maximal and properly contains no other prime ideals (2.4); hence, P^* is also maximal and X is a P-space. But $|\beta X \backslash X| \leqq 1$ (recall $\tilde{\theta}_X$) and so X is pseudo-compact [GJ, 6J]. The conclusion now follows as every pseudo-compact P-space is finite [GJ, 4K].

(b) Let ζX be the one-point compactification of its discrete subspace Y with $P^* \in \zeta X$ the point-at-infinity. Each point $P \in Y$ is fixed maximal and $\tilde{\theta}_X(P)$ is a zero-set point of X (2.3(c)). If P^* is also maximal, then the conclusion follows as in (a); we claim that this must be the case. Suppose that $\tilde{\theta}_X(P^*) = x \in X$ and let $X' = X \backslash \{x\}$; $X' \neq \varnothing$ as X has at least two points. Then X is the one-point compactification ($X = \tilde{\theta}_X[\zeta X]$) of X' which is a P-space [GJ, 4L]. Moreover, X' is a cozero-set of X (as $M_x \in Y$) and so X' is countable and discrete (as in the proof of 2.3); that is, $X = N^*$, the one-point compactification of N with x the point-at-infinity. But then the cardinality of the set of prime z-ideals properly contained in M_x is $|\beta N \backslash N|$ [GJ, 14G]; a contradiction as this set contains only P^*. □

3. **Characterizations of ζX and its place between βX_P and βX.** Recall that the $H_{g,h}=\mathscr{k}(g)\cap\mathscr{k}^c(h)$ $(g, h\in C(X))$ form a base for ζX; this base actually distinguishes ζX from other compactifications of X_P.

3.1 LEMMA. *Let* $g, h\in C(X)$. *Then* $\mathscr{k}(g)=\mathrm{cl}_{\zeta X}Z(g)$ *and* $\mathscr{k}^c(g)=\mathrm{cl}_{\zeta X}\operatorname{coz}(g)$; hence, $H_{g,h}=\mathrm{cl}_{\zeta X}Z(g)\cap\mathrm{cl}_{\zeta X}\operatorname{coz}(h)$.

Proof. It is easy to verify in general that if A is a dense subset of a topological space Y and U is open in Y, then $U\subseteq\mathrm{cl}_Y(A\cap U)$ [K, 1G]. Apply this to $A=X_P$, $Y=\zeta X$, $U_1=\mathscr{k}(g)$ and $U_2=\mathscr{k}^c(g)$ to obtain one pair of inclusions; the reverse inclusions are obvious. □

3.2 THEOREM (CHARACTERIZATION OF ζX). *If Y is a compactification of X_P and the collection* $\{\mathrm{cl}_Y Z(g)\cap\mathrm{cl}_Y\operatorname{coz}(h): g, h\in C(X)\}$ *is a base for Y, then there is a homeomorphism f of Y onto ζX which leaves X_P pointwise fixed.*

Observe that any such Y is totally-disconnected (having a base consisting of closed sets) and any such homeomorphism is unique (as X_P is dense).

Proof. Let Y be such a compactification; for $g, h\in C(X)$ we let $B_{g,h}=\mathrm{cl}_Y Z(g)\cap\mathrm{cl}_Y\operatorname{coz}(h)$.

(a) If $x\in X_P$, then $x\in B_{g,h}$ if and only if $x\in H_{g,h}$. Hence, $B_{g_1,h_1}\cap\cdots\cap B_{g_n,h_n}=\varnothing$ if and only if $H_{g_1,h_1}\cap\cdots\cap H_{g_n,h_n}=\varnothing$.

Proof of (a). $X_P\cap\mathrm{cl}_Y Z(g)=Z(g)=X_P\cap\mathrm{cl}_{\zeta X}Z(g)$; likewise for $\operatorname{coz}(h)$. The first statement is now clear and the second follows as X_P is dense in each of Y and ζX.

(b) If $y\in Y$, then $H_y=\bigcap_{y\in B_{g,h}}H_{g,h}$ is a singleton in ζX.

Proof of (b). The collection $\{H_{g,h}: y\in B_{g,h}\}$ of closed sets in compact ζX has the finite-intersection-property (by (a)) and so $H_y\neq\varnothing$. Suppose $P\neq Q\in H_y$; then there exist $g_1, h_1\in C(X)$ with $P\in H_{g_1,h_1}$ and $Q\notin H_{g_1,h_1}$. Then $y\notin B_{g_1,h_1}$ and so there is B_{g_2,h_2} with $y\in B_{g_2,h_2}$ and $B_{g_1,h_1}\cap B_{g_2,h_2}=\varnothing$; a contradiction as $P\in H_{g_1,h_1}\cap H_{g_2,h_2}$.

We now have the function $f: Y\to\zeta X$ defined by $\{f(y)\}=H_y$. Precisely the same argument exhibits a function $f^{-1}:\zeta X\to Y$ with $\{f^{-1}(P)\}=\bigcap_{P\in H_{g,h}}B_{g,h}$. Observing that $y\in B_{g,h}$ implies $f(y)\in H_{g,h}$ and $P\in H_{g,h}$ implies $f^{-1}(P)\in B_{g,h}$, one sees that $f\circ f^{-1}=1_{\zeta X}$, $f^{-1}\circ f=1_Y$, $f^{\leftarrow}[H_{g,h}]=B_{g,h}$ and $f[B_{g,h}]=H_{g,h}$. □

The above argument actually shows that any $X\in\mathscr{T}_{3\frac{1}{2}}$ which has a base $\mathscr{B}$ consisting of closed sets has at most one compactification Y with the collection $\{\mathrm{cl}_Y B_1\cap\mathrm{cl}_Y(X\backslash B_2): B_1, B_2\in\mathscr{B}\}$ as a base.

Since βX_P is a known compactification of X_P, the Characterization Theorem leads one to consider the collection $\mathscr{V}=\{V_{g,h} : g, h\in C(X)\}$, where $V_{g,h}=\mathrm{cl}_{\beta X_P}Z(g)\cap\mathrm{cl}_{\beta X_P}\operatorname{coz}(h)$. A straightforward calculation (using 2.2 and [GJ, 6.9(c)]) shows that $(\bar{\nu}_X)^{\leftarrow}[H_{g,h}]=V_{g,h}$, where $\bar{\nu}_X$ denotes, as usual, the Stone-extension of $\nu_X: X_P\to\zeta X$. Thus $\mathscr{V}$ is a base for the weak topology τ on the set βX_P generated by $\bar{\nu}_X$; also, $(\beta X_P, \tau)$ is coarser than βX_P and so is compact. Recalling that $\bar{\nu}_X$ is onto ζX (as X_P is dense in ζX), it follows that ζX is the T_0-quotient of $(\beta X_P, \tau)$.

We have already noted (2.2) that ζX is the quotient space of βX_P under the equivalence relation: $M_1 \sim M_2 \Leftrightarrow Z[M_1] \cap Z(X) = Z[M_2] \cap Z(X)$. Consequently: $\beta X_P \cong \zeta X$ iff $(\beta X_P, \tau) \cong \zeta X$ iff $Z[M_1] \cap Z(X) = Z[M_2] \cap Z(X) \Rightarrow M_1 = M_2$ (where $\cong$ indicates a canonical homeomorphism, that is, one which leaves X_P pointwise fixed). Observe that for the space R of reals, $R_P = R_D$, $|\beta R_P| = 2^{2^c}$ and $|\zeta R| = 2^c$.

4. **The natural transformations $\tilde{\theta}$ and θ, and the push-out β.** We have already noted the continuity of $\tilde{\theta}_X: K_{C(X)} \to \beta X$; let θ_X denote the restriction of $\tilde{\theta}_X$ to ζX.

4.1 Proposition. *$\tilde{\theta} = \{\tilde{\theta}_X\}_{X \in \mathscr{T}_{3\frac{1}{2}}}$ is a natural transformation from K_C to β; hence, $\theta = \{\theta_X\}_{X \in \mathscr{T}_{3\frac{1}{2}}}$ is a natural transformation from ζ to β.*

Proof. We must show that $\tilde{\theta}_X \circ K_{C(f)} = f \circ \tilde{\theta}_Y$ if $f: Y \to X$. Suppose not, then there is a $P \in K_{C(Y)}$ and $g \in C(X)$ with $g \circ f \in P$, but $\bar{f}(p) \notin \mathrm{cl}_{\beta X} Z(g)$, where $p = \tilde{\theta}_Y(P)$. But then $\bar{f}^{\leftarrow}[\beta X \backslash \mathrm{cl}_{\beta X} Z(g)]$ is a neighborhood of p in βY which misses $Z(g \circ f)$; contrary to $p \in \mathrm{cl}_{\beta Y} Z(g \circ f)$. $\square$

A *push-out* in a category A is a commutative diagram $m_2 \circ m_1 = m_4 \circ m_3$ in A which is best possible in the lower right corner; that is, if $m_2' \circ m_1 = m_4' \circ m_3$, then there is a morphism m unique with the property that $m \circ m_2 = m_2'$ and $m \circ m_4 = m_4'$. Push-outs are clearly unique to within equivalences; that is, if $m_2' \circ m_1 = m_4' \circ m_3$ is also a push-out, then m is an equivalence.

Let e denote the natural transformation from the identity functor $I_{3\frac{1}{2}}: \mathscr{T}_{3\frac{1}{2}} \to \mathscr{T}_{3\frac{1}{2}}$ to the reflector functor $\beta: \mathscr{T}_{3\frac{1}{2}} \to \mathscr{C}$; that is, e_X is the embedding of X into βX. The natural transformations θ and ν are precisely those needed to show that $e \circ p = \theta \circ \nu$ is a push-out in Fun $[\mathscr{T}_{3\frac{1}{2}}, \mathscr{T}_{3\frac{1}{2}}]$, the category whose objects are functors from $\mathscr{T}_{3\frac{1}{2}}$ to $\mathscr{T}_{3\frac{1}{2}}$ and whose morphisms are natural transformations.

4.2 Theorem. *$e \circ p = \theta \circ \nu$ is a push-out in* Fun $[\mathscr{T}_{3\frac{1}{2}}, \mathscr{T}_{3\frac{1}{2}}]$.

Proof. It is straightforward to show that $e_X \circ p_X = \theta_X \circ \nu_X$ if $X \in \mathscr{T}_{3\frac{1}{2}}$. We let n.t. $[F, G]$ denote, as usual, the natural transformations from F to G. Suppose $F: \mathscr{T}_{3\frac{1}{2}} \to \mathscr{T}_{3\frac{1}{2}}$, $\eta \in$ n.t. $[I_{3\frac{1}{2}}, F]$ and $\lambda \in$ n.t. $[\zeta, F]$ are such that $\eta \circ p = \lambda \circ \nu$. We seek $\gamma \in$ n.t. $[\beta, F]$ such that $\gamma \circ \theta = \lambda$, $\gamma \circ e = \eta$ and whenever $\gamma' \in$ n.t. $[\beta, F]$ so commutes, then $\gamma' = \gamma$.

Let $X \in \mathscr{T}_{3\frac{1}{2}}$. Since $\eta_X \circ p_X = \lambda_X \circ \nu_X$, we have $\eta_X[X] \subseteq \lambda_X[\zeta X]$; hence, $\eta_X': x \mapsto \eta_X(x)$ $(x \in X)$ defines a continuous function $\eta_X': X \to \lambda_X[\zeta X]$. Since $\lambda_X[\zeta X]$ is compact, we have the Stone-extension $\bar{\eta}_X': \beta X \to \lambda_X[\zeta X]$; hence $\gamma_X: p \mapsto \bar{\eta}_X'(p)$ defines a continuous function $\gamma_X: \beta X \to FX$. We claim that $\gamma = \{\gamma_X\}_{X \in \mathscr{T}_{3\frac{1}{2}}}$ is our desired natural transformation from β to F.

To see that $\gamma \in$ n.t. $[\beta, F]$, let $f: Y \to X$. Since $\eta \in$ n.t. $[I_{3\frac{1}{2}}, F]$, we have $F(f) \circ \eta_Y = \eta_X \circ f$. With this equality in mind, the two maps $\gamma_X \circ \bar{f}$ and $F(f) \circ \gamma_Y$ are seen to agree on dense Y of βY and hence are equal.

The equality $\gamma \circ e = \eta$ is straightforward and the equality $\gamma \circ \theta = \lambda$ follows from the agreement of $\gamma_X \circ \theta_X$ with λ_X on dense $\mathscr{M}_F(X)$.

Finally, suppose that $\gamma' \in$ n.t. $[\beta, F]$ with $\gamma' \circ \theta = \lambda$ and $\gamma' \circ e = \eta$. Then $\gamma' = \gamma$ as γ'_X and γ_X agree on dense X of βX. $\square$

4.3 COROLLARY. *$e_X \circ p_X = \theta_X \circ \nu_X$ is a push-out in $\mathscr{T}_{3\frac{1}{2}}$.* $\square$

5. **The continuous and order-preserved images of ζY.** If Y^* is a compactification of Y and f: $Y^* \to X \in \mathscr{C}$, it is straightforward to show that $f[Y^*] = \mathrm{cl}_X f[Y]$; in particular, if g: $Y \to X$ in $\mathscr{T}_{3\frac{1}{2}}$, then $\bar{g}[\beta Y] = \mathrm{cl}_{\beta X}\, g[Y]$. Moreover, if the f above is also a homeomorphism, then $f[Y^* \setminus Y] = \mathrm{cl}_X f[Y] \setminus f[Y]$ [GJ, 6.11]. The following result is now clear.

5.1 PROPOSITION. *If f: $Y \to X$, then both ζX and βX contain a continuous and order-preserved image of ζY; namely, $\hat{f}[\zeta Y]$ and $\theta_X \circ \hat{f}[\zeta Y]$, respectively. Moreover*:

(a) $\hat{f}[\zeta Y] = \mathrm{cl}_{\zeta X} f[Y]$, *and if f is also a homeomorphism, then* $\hat{f}[\zeta Y \setminus Y_P] = \mathrm{cl}_{\zeta X} f[Y] \setminus f[Y]$.

(b) $\tilde{\theta}_X \circ K_{C(f)}[K_{C(Y)}] = \theta_X \circ \hat{f}[\zeta Y] = \mathrm{cl}_{\beta X} f[Y]$, *and if f is also a homeomorphism and Y is a P-space, then* $\theta_X \circ \hat{f}[\zeta Y \setminus Y_P] = \mathrm{cl}_{\beta X} f[Y] \setminus f[Y]$. $\square$

An alternate proof of the first set of equalities in (b) is provided by the fact that $\tilde{\theta} \in$ n.t. $[K_C, \beta]$ and $\theta \in$ n.t. $[\zeta, \beta]$.

5.2 COROLLARY [GJ, 14F].

(a) *A z-ideal P of $C(X)$ is prime if and only if there is an ultrafilter $\mathscr{U}$ on X such that $P = \{g \in C(X) : Z(g) \in \mathscr{U}\}$.*

(b) *The prime z-ideals contained in M^p are precisely the z-ideals P such that $P = \{g \in C(X) : Z(g) \in \mathscr{U}\}$ for some ultrafilter $\mathscr{U}$ on X that converges to p.*

Proof. The mapping $\mathscr{U} \mapsto \{g \in C(X) : Z(g) \in \mathscr{U}\}$ is the ζ-extension of the obvious map j: $X_D \to X_P$, where X_D is the set X discretely topologized; moreover, $\hat{j}$ is onto ζX (5.1(a)). Since an ultrafilter $\mathscr{U}$ on X converges to $p \in \beta X$ if and only if $p = \bar{j}(\mathscr{U})$ [GJ, 6.6(a)], statement (b) holds because $\theta \in$ n.t. $[\zeta, \beta]$. $\square$

Note that statements similar to those in 5.2 can be obtained by considering the ζ-extension of p_X: $X_P \to X$; in this connection, see 2.2.

It follows directly from 5.1 that $\hat{f}$ maps ζY onto ζX if and only if $f[Y]$ is dense in X_P; and that $\theta_X \circ \hat{f}$ maps ζY onto βX if and only if $f[Y]$ is dense in X. Before formally stating these conclusions, however, we introduce a concept which leads, among other things, to further equivalent conditions (see 5.4 and 7.1).

DEFINITION. Let $Y \subseteq X$. A subset S of $C(X)$ is *Y-unit-free*, or Y-u.f., if $g|_Y$ is not a unit of $C(Y)$ (equivalently, $Z(g) \cap Y \neq \varnothing$) for each $g \in S$. We let $\tilde{T}(Y) = \{P \in K_{C(X)} : P$ is Y-u.f.$\}$ and $T(Y) = \tilde{T}(Y) \cap \zeta X$.

Following is a list of properties, some obvious, which is included here for future reference; recall [GJ, 2.7] that $Z^{\leftarrow}(Z(P)) = \{g \in C(X) : Z(g) = Z(h)$ for some $h \in P\}$ is the smallest prime z-ideal containing $P \in K_{C(X)}$.

5.3 LEMMA.

(a) *Let $Y \subseteq X$, $S \subseteq C(X)$ and $P \in K_{C(X)}$.*

(1) *If S is Y-u.f. and $S' \subseteq S$, then S' is Y-u.f.*

(2) $P \in \tilde{T}(Y)$ *if and only if* $Z^{\leftarrow}(Z(P)) \in \tilde{T}(Y)$.

(3) $\tilde{T}(Y)$ *is closed in* $K_{C(X)}$; *hence*, $T(Y)$ *is closed in* ζX.

(b) *Let* $f\colon Y \to X$ *and* $S \subseteq C(X)$.

(1) S *is* $f[Y]$-*u.f. if and only if* $g \circ f$ *is not a unit of* $C(Y)$ *for each* $g \in S$.

(2) $K_{C(f)}[K_{C(Y)}] \subseteq \tilde{T}(f[Y])$; *hence*, $\hat{f}[\zeta Y] \subseteq T(f[Y])$.

(3) $\mathrm{cl}_{X_P} f[Y] = \hat{f}[\zeta Y] \cap X_P = T(f[Y]) \cap X_P$.

Proof. (a)(3): If $P \in K_{C(X)} \backslash \tilde{T}(Y)$, then there is a $g \in P$ with $Z(g) \cap Y = \varnothing$ and $\tilde{h}(g)$ is a neighborhood of P which misses $\tilde{T}(Y)$. In (b)(2) note that $g \circ f \in Q \in K_{C(Y)}$ implies $g \circ f$ is not a unit of $C(Y)$. Turning to (b)(3), the first equality is clear from 5.1(a), and the second set is contained in the third by (b)(2). Finally, the third is contained in the first as every basic neighborhood $Z(g)$ of a point x in $T(f[Y]) \cap X_P$ meets $f[Y]$ ($Z(g) \cap f[Y] \neq \varnothing$ as $g \in M_x \in T(f[Y])$). □

Since the G_δ-subsets of X also form a base for X_P [GJ, 1.10, 1.14(a) and 3.2(b)], it is convenient to refer to $\mathrm{cl}_{X_P} Y$ as the *G_δ-closure* of Y in X; similarly, *G_δ-closed* and *G_δ-dense* subsets of X are defined (this terminology is found in [CN, 5]). It is clear that Y is G_δ-dense in X if and only if $X \backslash Y$ contains no nonvoid zero-sets of X.

5.4 Proposition. *The following assertions are equivalent for* $f\colon Y \to X$: (a) $\hat{f}[\zeta Y] = \zeta X$; (b) $f[Y]$ is G_δ-dense in X; (c) $X \backslash f[Y]$ *contains no nonvoid zero-sets of* X; (d) $\tilde{T}(f[Y]) = K_{C(X)}$; (e) $T(f[Y]) = \zeta X$; (f) $X_P \subseteq \hat{f}[\zeta Y]$; (g) $X_P \subseteq T(f[Y])$.

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c): 5.1(a) and the remarks immediately above. (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e): Every nonunit is contained in some prime ideal, and 5.3(a)(2). (b) $\Leftrightarrow$ (e) and (b) $\Leftrightarrow$ (f): 5.3(b)(3). □

Since $h \mapsto h|_X$ is an isomorphism of $C(\beta X)$ onto $C^*(X)$, it is clear that $\hat{e}_X\colon \zeta X \cong \zeta(\beta X)$ if X is pseudo-compact; the converse is now seen to hold by 5.4 (a) $\Leftrightarrow$ (c) and [GJ, 6I] (we will see in 6.2 that $\hat{e}_X$ is always a homeomorphism).

Before proceeding, however, note that if $Y \subseteq X$, then 5.1(b), 5.2(b)(2) and 5.2(a)(1) show that

$$\mathrm{cl}_{\beta X}\, Y = \theta_X \circ \hat{\imath}[\zeta Y] \subseteq \theta_X[T(Y)] \subseteq \{p \in \beta X : O^p \text{ is } Y\text{-u.f.}\},$$

where $i\colon Y \to X$ is inclusion and, for $p \in \beta X$, $O^p = \{g \in C(X) : \mathrm{cl}_{\beta X} Z(g)$ is a neighborhood of $p\}$ [GJ, 7.12].

5.5 Proposition. *If* $Y \subseteq X$ *and* $i\colon Y \to X$ *is inclusion, then*

$$\mathrm{cl}_{\beta X}\, Y = \theta_X \circ \hat{\imath}[\zeta Y] = \theta_X[T(Y)] = \{p \in \beta X : O^p \text{ is } Y\text{-}u.f.\}.$$

Proof. If $p \in \beta X$ and O^p is Y-u.f., we show that $p \in \mathrm{cl}_{\beta X}\, Y$. Let U be a neighborhood of p; without loss of generality, $U = Z_{\beta X}(h)$ for some $h \in C(\beta X)$ [GJ, 3.2(b)]. Then $h|_X \in O^p$ [GJ, 7.12(a)] and so $Z(h|_X)$ meets Y as O^p is Y-u.f.; note that $Z(h|_X) \subseteq U$. □

6. $\hat{f}$ **and** z**-embedding.** Let $f\colon Y\to X$ be given. Since ζY is compact and $\hat{f}$ preserves order, we have: $\hat{f}$ is an (into) order-isomorphism iff $\hat{f}$ is one-to-one iff $\hat{f}$ is a (into) homeomorphism. A sufficient condition for $\hat{f}$ to be one-to-one is that $\hat{f}$ be a z-embedding; it is not known if this is also a necessary condition.

DEFINITION. A subset Y of X is *z-embedded* in X if $Z(Y)\subseteq\{Z\cap Y : Z\in Z(X)\}$; also, $f\colon Y\to X$ is a *z-embedding* if f is a homeomorphism whose image is z-embedded.

Although a C^*-embedded subset is clearly z-embedded, the converse does not generally hold as every z-embedded subset of X is C^*-embedded if and only if X is an F-space [H]. Other subsets which are always z-embedded are cozero-sets [GJ_1, 3.1] and Lindelöf subspaces (Jerison, [HJ_1, 5.3]).

For $f\colon Y\to X$ and $P\in\zeta X$ we let P_f denote the subset $P_f=\{h\in C(Y) : Z(h)=Z(g\circ f)$ for some $g\in P\}$ of $C(Y)$. Clearly, $P\subseteq\{g\in C(X)\colon g\circ f\in P_f\}$, the latter being $\hat{f}(P_f)$ if $P_f\in\zeta Y$. But P_f is not generally a point of ζY; if it is, then $P\in T(f[Y])$ (as $P\subseteq\hat{f}(P_f)$ is $f[Y]$-u.f.) and we have seen (5.4) that $\zeta X=T(f[Y])$ if and only if $f[Y]$ is G_δ-dense in X. For $P\in\zeta X$ it is easy to verify that $P=\{g\in C(X) : g\circ f\in P_f\}$ if and only if P has property:

6.1 $h\in C(X)$, $g\in P$ and $Z(h\circ f) = Z(g\circ f)$ imply $h\in P$.

As its proof indicates, the following theorem is essentially due to Mark Mandelker.

6.2 THEOREM.

(a) *$f\colon Y\to X$ is a z-embedding if and only if the mapping of $Z(X)$ into $Z(Y)$ defined by $Z\mapsto f^{\leftarrow}[Z]$ is onto.*

(b) *Let $f\colon Y\to X$ be a z-embedding.*

(1) *$\hat{f}$ is an order-isomorphism and hence a homeomorphism.*

(2) *If $P\in\zeta X$, then $P_f\in\zeta Y$ if and only if $P\in T(f[Y])$. In fact, if $P\in T(f[Y])$, then P_f is the smallest prime z-ideal of $C(Y)$ with $P\subseteq\hat{f}(P_f)$; and if $P\in\hat{f}[\zeta Y]$, then P_f is the unique point of ζY with $P=\hat{f}(P_f)$.*

(3) *$\hat{f}[\zeta Y]=\{P\in T(f[Y]) : P$ has property 6.1$\}$ and $T(f[Y])=\{P\in\zeta X : P\subseteq\hat{f}(M^p)$ for some $p\in\beta Y\}$.*

(4) *$\hat{f}[\hbar(O^p)]=\{P\in\zeta X : P\subseteq\hat{f}(M^p)\}$* — *$\hat{f}[\zeta Y]=T(f[Y])$ if and only if the equality $\hat{f}[\hbar(O^p)]=\{P\in\zeta X : P\subseteq\hat{f}(M^p)\}$ holds for each $p\in\beta Y$.*

Proof. Statement (a) is [M_2, Theorem II] and shows that the proofs given in [M, I.5] generalize to obtain (b)(1) and (b)(2) with the exception of the implication $P_f\in\zeta Y\Rightarrow P\in T(f[Y])$ which, however, is noted above. Parts (b)(3) and (b)(4) now follow from (b)(2); specifically, for (b)(3) observe the remark incorporating 6.1 and for (b)(4) recall that $\hat{f}$ preserves order. $\square$

For future reference, we single out the result most useful to us.

6.3 COROLLARY. *Let $f\colon Y\to X$ be a z-embedding such that $\hat{f}[\zeta Y]=T(f[Y])$. Then $\hat{f}$ is an order-isomorphism homeomorphism of ζY onto $\{P\in\zeta X : P\subseteq\hat{f}(M^p)$ for* *some* $p \in \beta Y\}$. *In fact*, $\hat{f}[\mathscr{h}(O^p)] = \{P \in \zeta X : P \subseteq \hat{f}(M^p)\} \subseteq \mathscr{h}(O^{f^*(p)})$ *if* $p \in \beta Y$, *with* $\hat{f}[\mathscr{h}(O_y)] = \mathscr{h}(O_{f(y)})$ *if* $y \in Y$.

Proof. Note that $\hat{f}(M^p) \subseteq M^{f^*(p)}$ for $p \in \beta Y$ (as $\theta \in$ n.t. $[\zeta, \beta]$) with $\hat{f}(M_y) = M_{f(y)}$ if $y \in Y$ (1.5). $\square$

Our interest is now directed toward finding z-embeddings with $\hat{f}[\zeta Y] = T(f[Y])$.

6.4 Proposition. *If* $f\colon Y \to X$ *has* $T(f[Y])$ *open in* ζX, *then* $\hat{f}[\zeta Y] = T(f[Y])$. *In particular*, $\hat{f}[\zeta Y] = T(f[Y])$ *if either of the following two conditions holds*:

(a) $X \backslash f[Y]$ *contains no nonvoid zero-sets of* X;

(b) $X \backslash f[Y]$ *is a zero-set of* X.

Proof. If $T(f[Y])$ is open, then so is $T(f[Y]) \backslash \hat{f}[\zeta Y] = T(f[Y]) \cap (\zeta X \backslash \hat{f}[Y])$; moreover, this set is contained in $\zeta X \backslash X_P$ (5.3(b)(3)) and so is empty as X_P is dense. If (a) holds, then $T(f[Y]) = \zeta X$ (5.4). Let (b) hold with $X \backslash f[Y] = Z(k)$; we show that $T(f[Y]) = \mathscr{h}^c(k)$. Since $Z(k) \cap f[Y] = \varnothing$, it is clear that $T(f[Y]) \subseteq \mathscr{h}^c(k)$. Suppose $P \in \mathscr{h}^c(k)$ and $P \notin T(f[Y])$; then there is $g \in P$ with $Z(g) \subseteq X \backslash f[Y] = Z(k)$, contrary to $k \notin P$. $\square$

6.5 Corollary [M_1, Theorem IIa]. *If* X *is locally-compact and* σ-*compact* (*equivalently*, $\beta X \backslash X$ *is a zero-set of* βX) *and* $p \in \beta X$, *then the collection of prime* z-*ideals of* $C(\beta X)$ *contained in* $\hat{e}_X(M^p)$ *is order-isomorphic to the collection of prime* z-*ideals of* $C(X)$ *contained in* M^p. $\square$

Observe that every pseudo-compact space provides a counterexample to the converse of 6.5 (see the remark immediately following 5.4).

7. **The space of minimal prime ideals.** The space $\mathscr{m}(A)$ of minimal prime ideals of $A \in \mathscr{R}$ topologized relative to Spec A has been studied extensively; see [K_1], [K_2] and [HJ]. It is clear that the topology on the set $\mathscr{m}(A)$ relative to K_A is finer than $\mathscr{m}(A)$; in fact, if A contains no nonzero nilpotent elements (in particular, if $A = C(X)$), then each $\tilde{\mathscr{h}}(a) \cap \mathscr{m}(A)$ is open in $\mathscr{m}(A)$ [HJ, 2.3] and these relative topologies coincide. We let $m(X)$ denote the space $\mathscr{m}(C(X))$ and note that $m(X) \subseteq \zeta X$ [GJ, 14.7].

It is easy to verify [HJ, 5.3] that no proper closed subset of $m(X)$ is mapped onto βX by θ_X.

7.1 Proposition. *The following are equivalent for* $f\colon Y \to X$: (a) $\theta_X \circ \hat{f}[\zeta Y] = \beta X$; (b) $f[Y]$ *is dense in* X; (c) $m(X) \subseteq T(f[Y])$.

Proof. (a) $\Leftrightarrow$ (b): 5.1(b). (b) $\Rightarrow$ (c): $\theta_X[m(X)] = \beta X$ and $\theta_X[T(f[Y])] = \mathrm{cl}_{\beta X} f[Y]$ (5.5). (c) $\Rightarrow$ (b): We show that the closed (5.3(a)(3)) subset $T(f[Y]) \cap m(X)$ of $m(X)$ is mapped onto βX by θ_X. Let $p \in \beta X = \mathrm{cl}_{\beta X} f[Y] = \theta_X[T(f[Y])]$; thus, there is $P \in T(f[Y])$ with $P \subseteq M^p$. Choose $Q \in m(X)$ with $Q \subseteq P$, then $Q \in T(f[Y]) \cap m(X)$ (5.3(a)(1)) and $\theta_X(Q) = p$. $\square$

Note that if $m(X) \subseteq \hat{f}[\zeta Y]$, then $f[Y]$ is dense (5.3(b)(2) and 7.1).

A proof of the following theorem of Mandelker permits us to apply the preceding proposition and $\theta \in$ n.t. $[\zeta, \beta]$. Mandelker's statement, although formulated for the case $e_Y\colon Y \to \beta Y$, is equivalent to that given below (as Y is C^*-embedded in X if and only if $\bar{\imath}\colon \beta Y = \mathrm{cl}_{\beta X}\, Y$ [GJ, 6.9(a)]).

7.2 Theorem (Mandelker [M_1, Theorem I]). *Let Y be densely C^*-embedded in X, $p \in \beta Y$ and $P \in K_{C(X)}$ with $P \subseteq M^{\bar{\imath}(p)}$, where $i\colon Y \to X$ is inclusion. Then P is comparable to $\hat{\imath}(M^p)$. In fact, $P \subseteq \hat{\imath}(M^p)$ if and only if $P \in \tilde{T}(Y)$; while $\hat{\imath}(M^p) \subset P$ if and only if $P \notin \tilde{T}(Y)$.*

Proof. (a): $P \subseteq \hat{\imath}(M^p)$ implies $P \in \tilde{T}(Y)$ (5.3(b)(2) and 5.3(a)(1)). (b): If $P \in \tilde{T}(Y)$, then $P' = Z^{\leftarrow}(Z(P)) \in T(Y)$ (5.3(a)(2)) and $P \subseteq P' \subseteq \hat{\imath}(P'_i) \subseteq M^{\bar{\imath}(p)}$ (6.2(b)). Hence, $\bar{\imath}(p) = \theta_X(\hat{\imath}(P'_i)) = \bar{\imath}(\theta_Y(P'_i))$ (as $\theta \in$ n.t. $[\zeta, \beta]$) and $p = \theta_Y(P'_i)$ (as $\bar{\imath}$ is a homeomorphism, Y being C^*-embedded); that is, $P'_i \subseteq M^p$ and so $P \subseteq P' \subseteq \hat{\imath}(P'_i) \subseteq \hat{\imath}(M^p)$. (c): To conclude the proof we show that $P \notin \tilde{T}(Y)$ implies $\hat{\imath}(M^p) \subset P$. This is accomplished by showing P and $\hat{\imath}(M^p)$ to be comparable as then (a) will yield the desired proper inclusion. Comparability is shown by choosing a $Q \in m(X)$ with $Q \subseteq P$ and verifying that $Q \subseteq \hat{\imath}(M^p)$ (this technique is valid by [GJ, 14.8] and is used by Mandelker in his proof). Choosing such a Q, we have $Q \in \tilde{T}(Y)$ (7.1) and $Q \subseteq M^{\bar{\imath}(p)}$; consequently, $Q \subseteq \hat{\imath}(M^p)$ (by (b), replacing P with Q). □

Note that if Y is z-embedded but not C^*-embedded, then we can choose $p \neq q$ in βY with $\bar{\imath}(p) = \bar{\imath}(q)$; hence, $\hat{\imath}(M^q) \subseteq M^{\bar{\imath}(q)} = M^{\bar{\imath}(p)}$ (the first inclusion as $\theta \in$ n.t. $[\zeta, \beta]$) but $\hat{\imath}(M^q)$ is not comparable to $\hat{\imath}(M^p)$ (as $\hat{\imath}$ is an order-isomorphism).

7.3 Corollary. *Let Y be densely z-embedded in X. Then Y is C^*-embedded if and only if $p \in \beta Y$ and $P \subseteq M^{\bar{\imath}(p)}$ in $K_{C(X)}$ imply that P is comparable to $\hat{\imath}(M^p)$.* □

7.4 Proposition. *Let $f\colon Y \to X$.*

(a) $\theta_X \circ \hat{f}[m(Y)] = \mathrm{cl}_{\beta X} f[Y]$.

(b) *If $f[Y]$ is dense and $\hat{f}[m(Y)] \subseteq m(X)$, then $\hat{f}[m(Y)] = m(X)$.*

Proof. (a): 5.1(b) provides the inclusion $\theta_X \circ \hat{f}[m(Y)] \subseteq \mathrm{cl}_{\beta X} f[Y]$ (as $m(Y) \subseteq \zeta Y$) as well as the reverse inclusion (for $p = \theta_X \circ \hat{f}(P) \in \mathrm{cl}_{\beta X} f[Y]$ choose $Q \in m(Y)$ with $Q \subseteq P$). (b): The argument of (a) now shows that $\theta_X[\hat{f}[\zeta Y] \cap m(X)] = \mathrm{cl}_{\beta X} f[Y]$ (as $\hat{f}[m(Y)] \subseteq m(X)$; i.e., $\hat{f}(Q) \in m(X)$ for the Q chosen in (a)) which is now all of βX; hence, $m(X) \subseteq \hat{f}[\zeta Y]$ (as no proper closed subset of $m(X)$ maps onto βX). □

To prove the following lemma we use: $P \in K_{C(X)}$ is in $m(X)$ if and only if $g \in P$ implies the existence of $h \notin P$ with $g \cdot h = 0$ [HJ, 1.1].

7.5 Lemma. *Let $f\colon Y \to X$ be a z-embedding. Then $\hat{f}[m(Y)] \subseteq m(X)$ if either one of the following two conditions holds:*

(a) *$f[Y]$ is dense;*

(b) *$f[Y]$ is a cozero-set of X.*

Proof. Let $Q \in m(Y)$ and $g \in \hat{f}(Q)$; we seek $g' \notin \hat{f}(Q)$ with $g \cdot g' = 0$. Choose $h \notin Q$ with $h \cdot (g \circ f) = 0$ (as $g \circ f \in Q \in m(Y)$) and choose $t \in C(X)$ with $Z(h)$

$=Z(t\circ f)$ (by 6.2(a)). Note that $t\notin \hat{f}(Q)$ (as $h\notin Q$) and $f[Y]\subseteq Z(t\cdot g)$ (as $h\cdot(g\circ f)=0$). Thus, if $f[Y]$ is dense we can let $g'=t$; and if $X\backslash f[Y]=Z(k)$ we can let $g'=k\cdot t$. □

7.6 Theorem. *If f: $Y\to X$ is a dense z-embedding, then $\hat{f}$ restricted to $m(Y)$ is a homeomorphism of $m(Y)$ onto $m(X)$.*

Proof. 6.2(b)(1), 7.4(b) and 7.5. □

Observe that denseness cannot be dropped from the hypothesis of this theorem (7.1 (b) $\Leftrightarrow$ (c) and 5.3(b)(2)). Also note that f: $Y\to X$ can be onto without $\hat{f}$: $m(Y)\cong m(X)$; for example, let $X\neq X_P$ and $f=p_X$: $X_P\to X$.

7.7 Corollary.

(a) [HJ, 5.2] *$\hat{e}_X$ restricted to $m(X)$ is a homeomorphism of $m(X)$ onto $m(\beta X)$.*

(b) *$\hat{\imath}$ restricted to $m(Q)$ is a homeomorphism of $m(Q)$ onto $m(R)$, where i denotes the inclusion of the rational numbers Q into R.*

(c) *If X is countable, then $\hat{\nu}_X$: $\beta X_D\cong m(\zeta X)$.*

Proof. Both i and ν_X are z-embeddings as countable spaces are Lindelöf. In (c), note that a countable P-space is discrete [GJ, 4K]. □

8. **Some final applications.** Suppose a topological space X has a subset A which is closed, contains only isolated points of X, and is disjoint from a subset $A'\subseteq X$ whose union with A is dense in X. Then, it is straightforward to verify that $\mathrm{cl}_X A'=X\backslash A$.

8.1 Theorem. *Let f: $Y\to X$ be such that $A=X\backslash f[Y]$ is finite and contains only zero-set points of X.*

(a) *$\hat{f}[\zeta Y]=\zeta X\backslash\{M_a: a\in A\}$.*

(b) *If f is a homeomorphism, then f is a z-embedding with $\hat{f}[\zeta Y]=T(f[Y])$ (so that the hypothesis of* 6.3 *is satisfied); also $\hat{f}[\zeta Y\backslash Y_P]=\zeta X\backslash X_P$.*

(c) *If f is a homeomorphism and $\mathrm{cl}_{\beta X} f[Y]\subseteq X$, then the collection of free maximal ideals of $C(Y)$ is mapped one-to-one onto the collection of prime z-ideals maximal among those properly contained in the M_a ($a\in A$).*

(d) *If f is a homeomorphism and no point of A is isolated in X, then $\hat{f}$ restricted to $m(Y)$ is a homeomorphism onto $m(X)$.*

Proof. Letting $A'=f[Y]$, A and A', viewed as subsets of ζX, are as in the preceding remark (2.3(c) and X_P is dense); hence, (a) holds by 5.1(a). Observe that $X\backslash f[Y]$ is a zero-set of X; hence, (b) holds by 6.4(b) and 5.1(a), respectively. In (c), since $\bar{f}(p)\in \mathrm{cl}_{\beta X} f[Y]\backslash f[Y]$ (by the opening remarks of §5), the conclusion follows from (a) and 6.3. Finally, in (d), $f[Y]$ is now dense in X and 7.6 applies. □

Observe that if the hypotheses of both (c) and (d) are satisfied, then X must be compact.

We now have a rather complete description of $\zeta([0, 1])$ in terms of ζR by taking a homeomorphism f: $R\to[0, 1]$ with $f[R]=(0, 1)$ and observing that all the various hypotheses of 8.1 are satisfied. Although the two isolated points M_0 and M_1 can be topologically absorbed by $\zeta([0, 1])$ (which contains uncountably many such points) to obtain $\zeta R \cong \zeta([0, 1])$, one loses the order-isomorphism properties of $\hat{f}$ in doing so.

Likewise, the inclusion i of N into $N^* = N \cup \{\omega\}$, its one-point compactification, yields the known [GJ, 14G] prime z-ideal structure of ζN^* in terms of βN; here, the minimalness of the prime z-ideals properly contained in M_ω follows from the noncomparability of the (maximal) prime ideals in $\zeta N \backslash N = \beta N \backslash N$ (or from 7.6).

Finally, insight is gained whenever one has a one-to-one sequence $s: N \to X$ such that $S = s[N]$ contains no cluster points of S in X (equivalently, s is a homeomorphism), for then: (1) $\theta_X \circ \hat{s}[\beta N] = \mathrm{cl}_{\beta X} S$ and $\theta_X \circ \hat{s}[\beta N \backslash N] = \mathrm{cl}_{\beta X} S \backslash S$ (5.1(b)); and (2) $\hat{s}$ is an order-isomorphism homeomorphism (S is Lindelöf and 6.2).

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